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The evaluation of matrix elements for non-canonical Weyl tableau basis states adapted to $U(n_1 + n_2) \supset U(n_1) \times U(n_2)$

III. Recoupling procedures of the unitary group for double tensor operators

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Summary. In this paper, the recoupling procedure in the unitary group for double tensor operators is presented using the embedding for the three group chains $U(n = n_1 + n_2) \supset U(n_1) \times U(n_2)$; $U(n_1 + 2) \supset U(n_1 + 1) \supset U(n_1)$; $U(n_2 + 2) \supset U(n_2 + 1) \supset U(n_2)$. It is a new algorithm for the calculation of matrix elements of $U(n)$ generator products in partitioned bases.

Key words: Unitary group approach – Recoupling procedures – Weyl tableau basis states

1. Introduction

Since its introduction for many-electron systems [1–3], the application of the unitary group approach (UGA) in quantum chemistry has been developed more and more. The UGA formalism has been employed in configuration interaction (CI) calculations, electron propagator theory, the coupled cluster approach, many-body perturbation theory, and MC SCF approaches [4–10]. The UGA developments were also extended to the case of more general, more than two-column irreducible representations (IRs) [11, 12].

Recently, an efficient partitioning [13–15] was developed:

$$U(n = n_1 + n_2) \supset U(n_1) \times U(n_2).$$

The number of basis function can be dramatically decreased thereby. Furthermore, Li and Paldus [16] developed and summarized the unitary group tensor operator algebras for many-electron systems, in which standard Clebsch–Gordan coefficients and isoscalar factors for the unitary group were extended to transformation coefficients and corresponding isoscalar factors relating Gelfand bases to partitioned bases. Then, the calculation of the matrix elements between partitioned bases can be reduced by a "segment" method.

Lin [15] achieved another "global" insight by the derivation of generalized coupling coefficients relating the matrix elements between canonical bases to those between non-canonical bases. It is the aim of this paper to extend the results of Ref. [15] to two-body operator matrix elements

$$\left\langle \begin{matrix}[V];[V_1'][V_2']\\ W_1',W_2'\end{matrix}\right| E_2E_1 \left| \begin{matrix}[V];[V_1][V_2]\\ W_1,W_2\end{matrix}\right\rangle$$

by the recoupling procedure [17] together with the subgroup embedding technique.

Because of the Hermiticity relation:

$$E_{ij}^{+}=E_{ji},$$

it will be necessary to consider three types of two-body operators, namely:

$$\begin{array}{lll}\text{type} & \text{I} & E_2E_1=E_LE_L\\ & \text{II} & E_2E_1=E_RE_L\\ & \text{III} & E_2E_1=E_LE_R,\end{array}$$

where E_R refers to raising operators and E_L to lowering operators.

This paper is organized as follows: a short review is outlined in Sect. 2. In Sect. 3, the shifts are discussed. The detailed recoupling procedures are treated in Sect. 4 where the corresponding tables are presented, and finally, in Sect. 5 an example is given to illustrate how to use the given tables.

2. Short review for the case of the one-body operator

We assume the reader to be familiar with [15]. In this section, we just repeat some of its results. The same symbols as in [15] will be used here. To avoid ambiguities, the indices a, b refer to the orbitals of $U(n_1)$ and i, j refer to those of $U(n_2)$, namely:

$$\begin{gathered}1\leqslant a,b\leqslant n_1,\\ n_1+1\leqslant i,j\leqslant n_1+n_2=n.\end{gathered}\tag{2.1}$$

A superscript of single prime refers to the final state, and double prime to the intermediate state.

The conclusion in [15] was that, in the non-trivial cases, the matrix elements of the one-body operator E_{ia} between the non-canonical bases adapted to:

$$U(n=n_1+n_2)\supset U(n_1)\times U(n_2)\tag{2.2}$$

can be expressed as the product of $U(n_1+1)$ and $U(n_2+1)$ matrix elements, times a generalized recoupling factor A:

$$\begin{aligned}&\left\langle \begin{matrix}[V];[V_1'][V_2']\\ W_1',W_2'\end{matrix}\right| E_{ia} \left| \begin{matrix}[V];[V_1][V_2]\\ W_1,W_2\end{matrix}\right\rangle\\ &=A\cdot\left\langle \begin{matrix}[V_{n_1+1}]\\ [V_1']\\ W_1'\end{matrix}\right| E_{n_1+1,a} \left| \begin{matrix}[V_{n_1+1}]\\ [V_1]\\ W_1\end{matrix}\right\rangle\cdot\left\langle \begin{matrix}[V_{n_2+1}]\\ [V_2']\\ W_2'\end{matrix}\right| E_{i,n_1+n_2+1} \left| \begin{matrix}[V_{n_2+1}]\\ [V_2]\\ W_2\end{matrix}\right\rangle,\end{aligned}\tag{2.3}$$

where the groups $U(n_1+1)$ and $U(n_2+1)$ satisfy the following embedding condition:

$$\begin{gathered}U(n_1+1)\supset U(n_1)\\ U(n_2+1)\supset U(n_2).\end{gathered}\tag{2.4}$$

The generalized recoupling coefficient A in Eq. (2.3) is only dependent on the IRs in the group chains of Eqs. (2.2) and (2.4).

If $[V_1] \times [V_2]$ denotes the irreducible representations of $U(n_1) \times U(n_2)$, and the explicit labels $[\lambda_1^{(p)}, \lambda_2^{(p)}]$ are introduced instead of $[V_p]$, where $\lambda_q^{(p)}$ is the number of boxes in the qth column of the Young diagram $[V_p]$, then all the four shifting effects for the case of many-electron systems are as follows:

$$\begin{aligned}
&(1) \quad [\lambda_1^{(1)} - 1, \lambda_2^{(1)}] \times [\lambda_1^{(2)} + 1, \lambda_2^{(2)}] \\
&(2) \quad [\lambda_1^{(1)} - 1, \lambda_2^{(1)}] \times [\lambda_1^{(2)}, \lambda_2^{(2)} + 1] \\
&(3) \quad [\lambda_1^{(1)}, \lambda_2^{(1)} - 1] \times [\lambda_1^{(2)} + 1, \lambda_2^{(2)}] \\
&(4) \quad [\lambda_1^{(1)}, \lambda_2^{(1)} - 1] \times [\lambda_1^{(2)}, \lambda_2^{(2)} + 1].
\end{aligned} \tag{2.5}$$

The explicit formulae for the coefficient A in different shifts were derived in [15]. It should be mentioned that these A's correspond to a special choice of the IRs of $[V_{n_1+1}]$ and $[V_{n_2+1}]$, namely:

$$\begin{aligned}
&[V_{n_1+1}] = [V_1'] + [1, 1], \\
&[V_{n_2+1}] = [V_2'].
\end{aligned} \tag{2.6}$$

It is also allowed to use other choices of IRs, and therefore different sets for A's may be obtained.

3. Discussion of the shifts caused by the two-body operator E_2E_1

From Sect. 2 it is known that there are four cases of shift for a one-body operator acting on a non-canonical basis

$$\left| \begin{matrix} [V]; [V_1][V_2] \\ W_1, W_2 \end{matrix} \right\rangle .$$

Obviously, there may be $4 \times 4 = 16$ cases of shift for two one-body operators acting in succession. Therefore a double index pq ($p, q = 1, \ldots, 4$) can be used to label the 16 different shift cases, where p refers to the shift caused by E_2, and q refers to E_1. On the other hand, however, there may be three different kinds of shift in each subgroup IR after the action of E_2E_1. For example, for type I (E_LE_L) the three possible shifts for $[V_1]$ are:

$$\begin{aligned}
&(\alpha) \quad [\lambda_1^{(1)}, \lambda_2^{(1)} - 2] \\
&(\beta) \quad [\lambda_1^{(1)} - 1, \lambda_2^{(1)} - 1] \\
&(\gamma) \quad [\lambda_1^{(1)} - 2, \lambda_2^{(1)}]
\end{aligned} \tag{3.1}$$

and for $[V_2]$

$$\begin{aligned}
&(\alpha') \quad [\lambda_1^{(2)}, \lambda_2^{(2)} + 2] \\
&(\beta') \quad [\lambda_1^{(2)} + 1, \lambda_2^{(2)} + 1] \\
&(\gamma') \quad [\lambda_1^{(2)} + 2, \lambda_2^{(2)}].
\end{aligned} \tag{3.2}$$

Table 1. The definitions of shift cases pq

Type I ($E_L E_L$)

p	$[V_1'] \times [V_2']$
1	$[(\lambda_1^{(1)})'' - 1, (\lambda_2^{(1)})''] \times [(\lambda_1^{(2)})'' + 1, (\lambda_2^{(2)})'']$
2	$[(\lambda_1^{(1)})'' - 1, (\lambda_2^{(1)})''] \times [(\lambda_1^{(2)})'', (\lambda_2^{(2)})'' + 1]$
3	$[(\lambda_1^{(1)})'', (\lambda_2^{(1)})'' - 1] \times [(\lambda_1^{(2)})'' + 1, (\lambda_2^{(2)})'']$
4	$[(\lambda_1^{(1)})'', (\lambda_2^{(1)})'' - 1] \times [(\lambda_1^{(2)})'', (\lambda_2^{(2)})'' + 1]$
q	$[V_1''] \times [V_2'']$
1	$[\lambda_1^{(1)} - 1, \lambda_2^{(1)}] \times [\lambda_1^{(2)} + 1, \lambda_2^{(2)}]$
2	$[\lambda_1^{(1)} - 1, \lambda_2^{(1)}] \times [\lambda_1^{(2)}, \lambda_2^{(2)} + 1]$
3	$[\lambda_1^{(1)}, \lambda_2^{(1)} - 1] \times [\lambda_1^{(2)} + 1, \lambda_2^{(2)}]$
4	$[\lambda_1^{(1)}, \lambda_2^{(1)} - 1] \times [\lambda_1^{(2)}, \lambda_2^{(2)} + 1]$

Type II ($E_R E_L$)

p	$[V_1'] \times [V_2']$
1	$[(\lambda_1^{(1)})'' + 1, (\lambda_2^{(1)})''] \times [(\lambda_1^{(2)})'' - 1, (\lambda_2^{(2)})'']$
2	$[(\lambda_1^{(1)})'' + 1, (\lambda_2^{(1)})''] \times [(\lambda_1^{(2)})'', (\lambda_2^{(2)})'' - 1]$
3	$[(\lambda_1^{(1)})'', (\lambda_2^{(1)})'' + 1] \times [(\lambda_1^{(2)})'' - 1, (\lambda_2^{(2)})'']$
4	$[(\lambda_1^{(1)})'', (\lambda_2^{(1)})'' + 1] \times [(\lambda_1^{(2)})'', (\lambda_2^{(2)})'' - 1]$
q	$[V_1''] \times [V_2'']$
1	$[\lambda_1^{(1)} - 1, \lambda_2^{(1)}] \times [\lambda_1^{(2)} + 1, \lambda_2^{(2)}]$
2	$[\lambda_1^{(1)} - 1, \lambda_2^{(1)}] \times [\lambda_1^{(2)}, \lambda_2^{(2)} + 1]$
3	$[\lambda_1^{(1)}, \lambda_2^{(1)} - 1] \times [\lambda_1^{(2)} + 1, \lambda_2^{(2)}]$
4	$[\lambda_1^{(1)}, \lambda_2^{(1)} - 1] \times [\lambda_1^{(2)}, \lambda_2^{(2)} + 1]$

Type III ($E_L E_R$)

p	$[V_1'] \times [V_2']$
1	$[(\lambda_1^{(1)})'' - 1, (\lambda_2^{(1)})''] \times [(\lambda_1^{(2)})'' + 1, (\lambda_2^{(2)})'']$
2	$[(\lambda_1^{(1)})'' - 1, (\lambda_2^{(1)})''] \times [(\lambda_1^{(2)})'', (\lambda_2^{(2)})'' + 1]$
3	$[(\lambda_1^{(1)})'', (\lambda_2^{(1)})'' - 1] \times [(\lambda_1^{(2)})'' + 1, (\lambda_2^{(2)})'']$
4	$[(\lambda_1^{(1)})'', (\lambda_2^{(1)})'' - 1] \times [(\lambda_1^{(2)})'', (\lambda_2^{(2)})'' + 1]$
q	$[V_1''] \times [V_2''']$
1	$[\lambda_1^{(1)} + 1, \lambda_2^{(1)}] \times [\lambda_1^{(2)} - 1, \lambda_2^{(2)}]$
2	$[\lambda_1^{(1)} + 1, \lambda_2^{(1)}] \times [\lambda_1^{(2)}, \lambda_2^{(2)} - 1]$
3	$[\lambda_1^{(1)}, \lambda_2^{(1)} + 1] \times [\lambda_1^{(2)} - 1, \lambda_2^{(2)}]$
4	$[\lambda_1^{(1)}, \lambda_2^{(1)} + 1] \times [\lambda_1^{(2)}, \lambda_2^{(2)} - 1]$

Table 2. The definitions of shift kinds xy

Type I ($E_L E_L$)

x	$[V_1']$
α	$[\lambda_1^{(1)}, \lambda_2^{(1)} - 2]$
β	$[\lambda_1^{(1)} - 1, \lambda_2^{(1)} - 1]$
γ	$[\lambda_1^{(1)} - 2, \lambda_2^{(1)}]$
y	$[V_2']$
α'	$[\lambda_1^{(2)}, \lambda_2^{(2)} + 2]$
β'	$[\lambda_1^{(2)} + 1, \lambda_2^{(2)} + 1]$
γ'	$[\lambda_1^{(2)} + 2, \lambda_2^{(2)}]$

Type II ($E_R E_L$) and type III ($E_L E_R$)

x	$[V_1']$
α	$[\lambda_1^{(1)} + 1, \lambda_2^{(1)} - 1]$
β	$[\lambda_1^{(1)}, \lambda_2^{(1)}]$
γ	$[\lambda_1^{(1)} - 1, \lambda_2^{(1)} + 1]$
y	$[V_2']$
α'	$[\lambda_1^{(2)} + 1, \lambda_2^{(2)} - 1]$
β'	$[\lambda_1^{(2)}, \lambda_2^{(2)}]$
γ'	$[\lambda_1^{(2)} - 1, \lambda_2^{(2)} + 1]$

Table 3. The relation between pq and xy

Type I ($E_L E_L$)

	y:	α'	β'	γ'
x: α	$pq =$	44	34, 43	33
β		24, 42	14, 23, 32, 41	13, 31
γ		22	12, 21	11

Type II ($E_R E_L$)

	y:	α'	β'	γ'
x: α	$pq =$	23	13, 24	14
β		21, 43	11, 22, 33, 44	12, 34
γ		41	31, 42	32

Type III ($E_L E_R$)

	y:	α'	β'	γ'
x: α	$pq =$	32	31, 42	41
β		12, 34	11, 22, 33, 44	21, 43
γ		14	13, 24	23

So, there will be $3 \times 3 = 9$ kinds of shift caused by $E_2 E_1$ and it is also possible to use another double index xy ($x = \alpha, \beta, \gamma; y = \alpha', \beta', \gamma'$) to label the 9 kinds of shift, where x refers to the shift of $[V_1]$ and y refers to that of $[V_2]$.

Clearly, all the 16 cases of pq are included in the 9 kinds xy. In Tables 1 and 2 the explicit definitions of these pq and xy are given, respectively, and in Table 3 the relations between pq and xy are listed.

4. Details of the recoupling procedure

It was mentioned in [15] that the operators E_L and E_R transform as tensor operators. Hence the operators $E_2 E_1$ will transform as double tensor operators. In principle, for a system being coupled of two parts, the reduced matrix element of a double tensor operator can be factorized into a linear combination of products of the reduced matrix elements of the two subgroups, times a recoupling coefficient depending only on the IRs [17]. Therefore, it is easy to reach the following conclusion by using the Wigner–Eckart theorem repeatedly, as in [15]: the matrix element of a double tensor operator can be factorized into a linear combination of products of two subgroups' matrix elements, times a similar recoupling coefficient being called the generalized recoupling coefficient.

Because of the special situation in the group $U(n)$, the generators of the unitary group $U(n+1)$, $E_{n+1,t}$ ($t = 1, 2, \ldots, n$), constitute of a vector operator of $U(n)$, and similarly for the generators of $U(n+2)$, $E_{n+2,t}$ ($t = 1, 2, \ldots, n$). So the following three group chains have to be considered simultaneously:

$$U(n_1+n_2) \supset U(n_1) \times U(n_2),$$
$$U(n_1+2) \supset U(n_1+1) \supset U(n_1),$$
$$U(n_2+2) \supset U(n_2+1) \supset U(n_2).$$

The general conclusion mentioned above can be obtained by a suitable recoupling procedure together with the embedding technique. We show in this section that for the three types $E_L E_L$, $E_R E_L$ and $E_L E_R$, the following formulae hold respectively:

(1) For $E_2 E_1 = E_L E_L$, we have:

$$\left\langle \begin{matrix}[V];[V'_1][V'_2]\\ W'_1, W'_2\end{matrix} \middle| E_{ia}E_{jb} \middle| \begin{matrix}[V];[V_1][V_2]\\ W_1, W_2\end{matrix} \right\rangle_{xy}$$
$$= \sum_{pq\in xy} A'_{pq} \cdot \left\langle \begin{matrix}[V_{n_1+2}]\\ [V'_{n_1+1}]\\ [V'_1]\\ W'_1\end{matrix} \middle| E_{n_1+1,a}E_{n_1+2,b} \middle| \begin{matrix}[V_{n_1+2}]\\ [V_{n_1+1}]\\ [V_1]\\ W_1\end{matrix} \right\rangle_{pq}$$
$$\times \left\langle \begin{matrix}[V_{n_2+2}]\\ [V'_{n_2+1}]\\ [V'_2]\\ W'_2\end{matrix} \middle| E_{i,n_1+n_2+2}E_{j,n_1+n_2+1} \middle| \begin{matrix}[V_{n_2+2}]\\ [V_{n_2+1}]\\ [V_2]\\ W_2\end{matrix} \right\rangle_{pq}. \tag{4.1}$$

(2) For $E_2 E_1 = E_R E_L$, we have

$$\left\langle \begin{matrix}[V];[V'_1][V'_2]\\ W'_1, W'_2\end{matrix} \middle| E_{ai}E_{jb} \middle| \begin{matrix}[V];[V_1][V_2]\\ W_1, W_2\end{matrix} \right\rangle_{xy}$$
$$= \sum_{pq\in xy} A'_{pq} \cdot \left\langle \begin{matrix}[V_{n_1+1}]\\ [V'_1]\\ W'_1\end{matrix} \middle| E_{a,n_1+1}E_{n_1+1,b} \middle| \begin{matrix}[V_{n_1+1}]\\ [V_1]\\ W_1\end{matrix} \right\rangle_{pq}$$
$$\times \left\langle \begin{matrix}[V_{n_2+1}]\\ [V'_2]\\ W'_2\end{matrix} \middle| E_{n_1+n_2+1,i}E_{j,n_1+n_2+1} \middle| \begin{matrix}[V_{n_2+1}]\\ [V_2]\\ W_2\end{matrix} \right\rangle_{pq}. \tag{4.2}$$

(3) For $E_2 E_1 = E_L E_R$, we have

$$\left\langle \begin{matrix}[V];[V'_1][V'_2]\\ W'_1, W'_2\end{matrix} \middle| E_{ia}E_{bj} \middle| \begin{matrix}[V];[V_1][V_2]\\ W_1, W_2\end{matrix} \right\rangle_{xy}$$
$$= \sum_{pq\in xy} A'_{pq} \cdot \left\langle \begin{matrix}[V_{n_1+1}]\\ [V'_1]\\ W'_1\end{matrix} \middle| E_{n_1+1,a}E_{b,n_1+1} \middle| \begin{matrix}[V_{n_1+1}]\\ [V_1]\\ W_1\end{matrix} \right\rangle_{pq}$$
$$\times \left\langle \begin{matrix}[V_{n_2+1}]\\ [V'_2]\\ W'_2\end{matrix} \middle| E_{i,n_1+n_2+1}E_{n_1+n_2+1,j} \middle| \begin{matrix}[V_{n_2+1}]\\ [V_2]\\ W_2\end{matrix} \right\rangle_{pq}. \tag{4.3}$$

In Eqs. (4.1) to (4.3) the summation runs over all the cases pq in the given kind xy. All matrix elements on the right-hand side are "partial matrix elements" because the index pq limits the choices of the IR in the intermediate state $[V_1''] \times [V_2'']$ (see Table 1).

We now direct our attention to obtain Eqs. (4.1) to (4.3). Concerning Eq. (4.3), the two-body operator matrix element appearing on the left-hand side can be turned into a sum of products of one-body matrix elements by the relation:

$$\left\langle \begin{matrix}[V];[V_1'][V_2']\\ W_1', W_2'\end{matrix} \middle| E_{ia}E_{bj} \middle| \begin{matrix}[V];[V_1][V_2]\\ W_1, W_2\end{matrix}\right\rangle_{xy}$$
$$= \sum_{[V_1'']\times[V_2'']} \sum_{|W_1'',W_2''\rangle\in[V_1'']\times[V_2'']} \left\langle \begin{matrix}[V];[V_1'][V_2']\\ W_1', W_2'\end{matrix} \middle| E_{ia} \middle| \begin{matrix}[V];[V_1''][V_2'']\\ W_1'', W_2''\end{matrix}\right\rangle_p$$
$$\times \left\langle \begin{matrix}[V];[V_1''][V_2'']\\ W_1'', W_2''\end{matrix} \middle| E_{bj} \middle| \begin{matrix}[V];[V_1][V_2]\\ W_1, W_2\end{matrix}\right\rangle_q . \tag{4.4}$$

In Eq. (4.4) the first summation includes all the possible different IRs $[V_1''] \times [V_2'']$ for intermediate states, the second includes all the Weyl bases $|W_1'', W_2''\rangle$ belonging to the given IR. So, all matrix elements of operator E_{ia} under the second summation have the same index p of shift case, according to the relation between $[V_1'] \times [V_2']$ and $[V_1''] \times [V_2'']$. Similarly, the matrix elements of operator E_{bj} have the same index q of shift case, according to the relation between $[V_1''] \times [V_2'']$ and $[V_1] \times [V_2]$. Then the two one-body operator matrix elements in Eq. (4.4) can be calculated according to [15] as:

$$\left\langle \begin{matrix}[V];[V_1'][V_2']\\ W_1', W_2'\end{matrix} \middle| E_{ia} \middle| \begin{matrix}[V];[V_1''][V_2'']\\ W_1'', W_2''\end{matrix}\right\rangle_p$$
$$= A_p \cdot \left\langle \begin{matrix}[V^p_{n_1+1}]\\ [V_1']\\ W_1'\end{matrix} \middle| E_{n_1+1,a} \middle| \begin{matrix}[V^p_{n_1+1}]\\ [V_1'']\\ W_1''\end{matrix}\right\rangle_p \cdot \left\langle \begin{matrix}[V^p_{n_2+1}]\\ [V_2']\\ W_2'\end{matrix} \middle| E_{i,n_1+n_2+1} \middle| \begin{matrix}[V^p_{n_2+1}]\\ [V_2'']\\ W_2''\end{matrix}\right\rangle_p , \tag{4.5}$$

where

$$\begin{aligned} &[V^p_{n_1+1}] \supset [V_1'], [V_1''] \\ &[V^p_{n_2+1}] \supset [V_2'], [V_2''], \end{aligned} \tag{4.6}$$

and

$$\left\langle \begin{matrix}[V];[V_1''][V_2'']\\ W_1'', W_2''\end{matrix} \middle| E_{bj} \middle| \begin{matrix}[V];[V_1][V_2]\\ W_1, W_2\end{matrix}\right\rangle_q$$
$$= A_q \cdot \left\langle \begin{matrix}[V^q_{n_1+1}]\\ [V_1'']\\ W_1''\end{matrix} \middle| E_{b,n_1+1} \middle| \begin{matrix}[V^q_{n_1+1}]\\ [V_1]\\ W_1\end{matrix}\right\rangle_q \cdot \left\langle \begin{matrix}[V^q_{n_2+1}]\\ [V_2'']\\ W_2''\end{matrix} \middle| E_{n_1+n_2+1,j} \middle| \begin{matrix}[V^q_{n_2+1}]\\ [V_2]\\ W_2\end{matrix}\right\rangle_q , \tag{4.7}$$

where

$$\begin{aligned} &[V^q_{n_1+1}] \supset [V_1''], [V_1] \\ &[V^q_{n_2+1}] \supset [V_2''], [V_2]. \end{aligned} \tag{4.8}$$

Substituting Eqs. (4.5), (4.7) into Eq. (4.4) and taking into account that A_p, A_q are only dependent on the IR, we get:

$$\left\langle \begin{matrix}[V];[V_1'][V_2'] \\ W_1', W_2'\end{matrix} \middle| E_{ia}E_{bj} \middle| \begin{matrix}[V];[V_1][V_2] \\ W_1, W_2\end{matrix} \right\rangle_{xy}$$

$$= \sum_{[V_1'']\times[V_2'']} A_pA_q \sum_{|W_1'', W_2''\rangle\in[V_1'']\times[V_2'']} \left\langle \begin{matrix}[V^p_{n_1+1}] \\ [V_1'] \\ W_1'\end{matrix} \middle| E_{n_1+1,a} \middle| \begin{matrix}[V^p_{n_1+1}] \\ [V_1''] \\ W_1''\end{matrix} \right\rangle_p$$

$$\times \left\langle \begin{matrix}[V^p_{n_2+1}] \\ [V_2'] \\ W_2'\end{matrix} \middle| E_{i,n_1+n_2+1} \middle| \begin{matrix}[V^p_{n_2+1}] \\ [V_2''] \\ W_2''\end{matrix} \right\rangle_p \left\langle \begin{matrix}[V^q_{n_1+1}] \\ [V_1''] \\ W_1''\end{matrix} \middle| E_{b,n_1+1} \middle| \begin{matrix}[V^q_{n_1+1}] \\ [V_1] \\ W_1\end{matrix} \right\rangle_q$$

$$\times \left\langle \begin{matrix}[V^q_{n_2+1}] \\ [V_2''] \\ W_2''\end{matrix} \middle| E_{n_1+n_2+1,j} \middle| \begin{matrix}[V^q_{n_2+1}] \\ [V_2] \\ W_2\end{matrix} \right\rangle_q$$

$$= \sum_{[V_1'']\times[V_2'']} A_pA_q \left[\sum_{|W_1''\rangle\in[V_1'']} \left\langle \begin{matrix}[V^p_{n_1+1}] \\ [V_1'] \\ W_1'\end{matrix} \middle| E_{n_1+1,a} \middle| \begin{matrix}[V^p_{n_1+1}] \\ [V_1''] \\ W_1''\end{matrix} \right\rangle_p \right.$$

$$\left. \times \left\langle \begin{matrix}[V^q_{n_1+1}] \\ [V_1''] \\ W_1''\end{matrix} \middle| E_{b,n_1+1} \middle| \begin{matrix}[V^q_{n_1+1}] \\ [V_1] \\ W_1\end{matrix} \right\rangle_q \right]$$

$$\times \left[\sum_{|W_2''\rangle\in[V_2'']} \left\langle \begin{matrix}[V^p_{n_2+1}] \\ [V_2'] \\ W_2'\end{matrix} \middle| E_{i,n_1+n_2+1} \middle| \begin{matrix}[V^p_{n_2+1}] \\ [V_2''] \\ W_2''\end{matrix} \right\rangle_p \right.$$

$$\left. \times \left\langle \begin{matrix}[V^q_{n_2+1}] \\ [V_2''] \\ W_2''\end{matrix} \middle| E_{n_1+n_2+1,j} \middle| \begin{matrix}[V^q_{n_2+1}] \\ [V_2] \\ W_2\end{matrix} \right\rangle_q \right]. \tag{4.9}$$

Obviously, Eq. (4.9) can be contracted further when a suitable choice of $[V_{n_1+1}]$ and $[V_{n_2+1}]$ satisfies:

$$[V_{n_1+1}]=[V^p_{n_1+1}]=[V^q_{n_1+1}] \quad \text{and} \quad [V_{n_2+1}]=[V^p_{n_2+1}]=[V^q_{n_2+1}]. \tag{4.10}$$

Furthermore, according to Eqs. (4.6) and (4.8), $[V_{n_1+1}]$ and $[V_{n_2+1}]$ should also satisfy:

$$[V_{n_1+1}] \supset [V_1'], [V_1''], [V_1] \quad \text{and} \quad [V_{n_2+1}] \supset [V_2'], [V_2''], [V_2]. \tag{4.11}$$

Clearly, such choices of $[V_{n_1+1}]$ and $[V_{n_2+1}]$ do really exist for all possible cases pq of this type of operator E_LE_R (later see Table 6). Therefore, Eq. (4.9) can be contracted to Eq. (4.3) with $A'_{pq}=A_pA_q$.

Equation (4.2) for type E_RE_L is obtained on a parallel way. As to Eq. (4.1) for type E_LE_L, however, this is a little more complicated, though we finally arrive at a similar equation:

$$\left\langle \begin{matrix}[V];[V_1'][V_2'] \\ W_1', W_2'\end{matrix} \right| E_{ia}E_{jb} \left| \begin{matrix}[V];[V_1][V_2] \\ W_1, W_2\end{matrix} \right\rangle_{xy}$$

$$= \sum_{[V_1'']\times[V_2'']} A_p A_q \left[\sum_{|W_1''\rangle\in[V_1'']} \left\langle \begin{matrix}[V^p_{n_1+1}] \\ [V_1'] \\ W_1'\end{matrix} \right| E_{n_1+1,a} \left| \begin{matrix}[V^p_{n_1+1}] \\ [V_1''] \\ W_1''\end{matrix} \right\rangle_p \right.$$

$$\left. \times \left\langle \begin{matrix}[V^q_{n_1+1}] \\ [V_1''] \\ W_1''\end{matrix} \right| E_{n_1+1,b} \left| \begin{matrix}[V^q_{n_1+1}] \\ [V_1] \\ W_1\end{matrix} \right\rangle_q \right]$$

$$\times \left[\sum_{|W_2''\rangle\in[V_2'']} \left\langle \begin{matrix}[V^p_{n_2+1}] \\ [V_2'] \\ W_2'\end{matrix} \right| E_{i,n_1+n_2+1} \left| \begin{matrix}[V^p_{n_2+1}] \\ [V_2''] \\ W_2''\end{matrix} \right\rangle_p \right.$$

$$\left. \times \left\langle \begin{matrix}[V^q_{n_2+1}] \\ [V_2''] \\ W_2''\end{matrix} \right| E_{j,n_1+n_2+1} \left| \begin{matrix}[V^q_{n_2+1}] \\ [V_2] \\ W_2\end{matrix} \right\rangle_q \right]. \tag{4.12}$$

This time one cannot find the desired $[V_{n_1+1}]$ and $[V_{n_2+1}]$ satisfying Eqs. (4.10) and (4.11). We illustrate this by the example $pq = 44$ corresponding to $xy = \alpha\alpha'$. In this case we have:

$$\begin{matrix} [V_1] = [\lambda_1^{(1)}, \lambda_2^{(1)}] & [V_1''] = [V_1] - [0,1] & [V_1'] = [V_1] - [0,2] \\ [V_2] = [\lambda_1^{(2)}, \lambda_2^{(2)}] & [V_2''] = [V_2] + [0,1] & [V_2'] = [V_2] + [0,2]. \end{matrix} \tag{4.13}$$

Therefore, the embedding of $U(n_1+1)$ and $U(n_2+1)$ into $U(n_1+2)$ and $U(n_2+2)$ has to be considered. Accordingly, Eq. (4.1) can be obtained provided one can "equivalently" transform the right-hand side of Eq. (4.12) into:

$$\sum_{[V_1'']\times[V_2'']} A_p A_q \left[\sum_{|W_1''\rangle\in[V_1'']} \left\langle \begin{matrix}[V_{n_1+2}] \\ [V'_{n_1+1}] \\ [V_1'] \\ W_1'\end{matrix} \right| E_{n_1+1,a} \left| \begin{matrix}[V_{n_1+2}] \\ [V''_{n_1+1}] \\ [V_1''] \\ W_1''\end{matrix} \right\rangle_p \right.$$

$$\left. \times \left\langle \begin{matrix}[V_{n_1+2}] \\ [V''_{n_1+1}] \\ [V_1''] \\ W_1''\end{matrix} \right| E_{n_1+2,b} \left| \begin{matrix}[V^q_{n_1+2}] \\ [V_{n_1+1}] \\ [V_1] \\ W_1\end{matrix} \right\rangle_q \right] \left[\sum_{|W_2''\rangle\in[V_2'']} \left\langle \begin{matrix}[V_{n_2+2}] \\ [V'_{n_2+1}] \\ [V_2'] \\ W_2'\end{matrix} \right| E_{i,n_1+n_2+2} \left| \begin{matrix}[V_{n_2+2}] \\ [V''_{n_2+1}] \\ [V_2''] \\ W_2''\end{matrix} \right\rangle_p \right.$$

$$\left. \times \left\langle \begin{matrix}[V_{n_2+2}] \\ [V''_{n_2+1}] \\ [V_2''] \\ W_2''\end{matrix} \right| E_{j,n_1+n_2+1} \left| \begin{matrix}[V_{n_2+2}] \\ [V_{n_2+1}] \\ [V_2] \\ W_2\end{matrix} \right\rangle_q \right] \tag{4.14}$$

where, instead of Eq. (4.11), the following requirements should hold:

$$
\begin{gathered}
[V_{n_1+2}] \supset [V'_{n_1+1}], [V''_{n_1+1}], [V_{n+1}] \\
[V'_{n_1+1}] \supset [V'_1], \qquad [V''_{n_1+1}] \supset [V''_1], \qquad [V_{n_1+1}] \supset [V_1] \\
[V_{n_2+2}] \supset [V'_{n_2+1}], [V''_{n_2+1}], [V_{n_2+1}] \\
[V'_{n_2+1}] \supset [V'_2], \qquad [V''_{n_2+1}] \supset [V''_2], \qquad [V_{n_2+1}] \supset [V_2].
\end{gathered}
\tag{4.15}
$$

The word "equivalently" means that the values of the four matrix elements in Eq. (4.14) are exactly equal to the four matrix elements on the right-hand side of Eq. (4.12).

One can see that the following choice is reasonable for the case of $pq = 44$:

$$
\begin{gathered}
[V_{n_1+2}] = [V_1] \\
[V'_{n_1+1}] = [V_1] - [0, 1] \qquad [V''_{n_1+1}] = [V_1] - [0, 1] \qquad [V_{n_1+1}] = [V_1] \\
[V'_1] = [V_1] - [0, 2] \qquad [V''_1] = [V_1] - [0, 1] \qquad [V_1] = [V_1] \\
[V_{n_2+2}] = [V_2] + [0, 2] \\
[V'_{n_2+1}] = [V_2] + [0, 2] \qquad [V''_{n_2+1}] = [V_2] + [0, 1] \qquad [V_{n_2+1}] = [V_2] + [0, 1] \\
[V'_2] = [V_2] + [0, 2] \qquad [V''_2] = [V_2] + [0, 1] \qquad [V_2] = [V_2].
\end{gathered}
$$

By this choice the above "equivalent" requirement holds for the following choice in Eq. (4.12):

$$
\begin{gathered}
[V^p_{n_1+1}] = [V_1] - [0, 1] \qquad [V^q_{n_1+1}] = [V_1] \\
[V^p_{n_2+1}] = [V_2] + [0, 2] \qquad [V^q_{n_2+1}] = [V_2] + [0, 1].
\end{gathered}
\tag{4.16}
$$

Reasonable choices for all the shifts in the three types are given in Tables 4 to 6, respectively.

The corresponding generalized recoupling coefficients $A'_{pq} = A_p A_q$ are obtained through a complicated and tedious derivation which is similar to that in [15] except for the different choice of Eq. (2.6). For the sake of simplification, we introduce to following formula:

$$
A'_{pq} = F_p B_q C_x D_y, \tag{4.17}
$$

where p, q, x and y are indices of the shifts. F_p, C_x depend on the parameters of the final state

$$
\left| \begin{matrix} [V]; [V'_1][V'_2] \\ W'_1 W'_2 \end{matrix} \right\rangle
$$

B_q, D_y depend on those of the initial state

$$
\left| \begin{matrix} [V]; [V_1][V_2] \\ W_1 W_2 \end{matrix} \right\rangle .
$$

Tables 7 and 8 give the explicit formulae.

Table 4. Reasonable choices of IRs for type I ($E_L E_L$)

	Level	Final	Intermediate	Initial
x: α	$U(n_1+2)$	$[V_1]$	$[V_1]$	$[V_1]$
	$U(n_1+2)$	$[V_1]-[0,1]$	$[V_1]-[0,1]$	$[V_1]$
	$U(n_1)$	$[V_1]-[0,2]$	$[V_1]-[0,1]_{q=3,4}$	$[V_1]$
β	$U(n_1+2)$	$[V_1]$	$[V_1]$	$[V_1]$
	$U(n_1+1)$	$[V_1]-[0,1]_{q=3,4}$	$[V_1]-[0,1]_{q=3,4}$	$[V_1]$
		$[V_1]-[1,0]_{q=1,2}$	$[V_1]-[1,0]_{q=1,2}$	
	$U(n_1)$	$[V_1]-[1,1]$	$[V_1]-[0,1]_{q=3,4}$	$[V_1]$
			$[V_1]-[1,0]_{q=1,2}$	
γ	$U(n_1+2)$	$[V_1]$	$[V_1]$	$[V_1]$
	$U(n_1+1)$	$[V_1]-[1,0]$	$[V_1]-[1,0]$	$[V_1]$
	$U(n_1)$	$[V_1]-[2,0]$	$[V_1]-[1,0]_{q=1,2}$	$[V_1]$
y: α'	$U(n_2+2)$	$[V_2]+[0,2]$	$[V_2]+[0,2]$	$[V_2]+[0,2]$
	$U(n_2+1)$	$[V_2]+[0,2]$	$[V_2]+[0,1]$	$[V_2]+[0,1]$
	$U(n_2)$	$[V_2]+[0,2]$	$[V_2]+[0,1]_{q=2,4}$	$[V_2]$
β'	$U(n_2+2)$	$[V_2]+[1,1]$	$[V_2]+[1,1]$	$[V_2]+[1,1]$
	$U(n_2+1)$	$[V_2]+[1,1]$	$[V_2]+[0,1]_{q=2,4}$	$[V_2]+[0,1]_{q=2,4}$
			$[V_2]+[1,0]_{q=1,3}$	$[V_2]+[1,0]_{q=1,3}$
	$U(n_2)$	$[V_2]+[1,1]$	$[V_2]+[0,1]_{q=2,4}$	$[V_2]$
			$[V_2]+[1,0]_{q=1,3}$	
γ'	$U(n_2+2)$	$[V_2]+[2,0]$	$[V_2]+[2,0]$	$[V_2]+[2,0]$
	$U(n_2+1)$	$[V_2]+[2,0]$	$[V_2]+[1,0]$	$[V_2]+[1,0]$
	$U(n_2)$	$[V_2]+[2,0]$	$[V_2]+[1,0]_{q,1,3}$	$[V_2]$

Table 5. Reasonable choices of IRs for type II ($E_R E_L$)

	Level	Final	Intermediate	Initial
x: α	$U(n_1+1)$	$[V_1]+[1,0]$	$[V_1]+[1,0]$	$[V_1]+[1,0]$
	$U(n_1)$	$[V_1]+[1,-1]$	$[V_1]-[0,1]_{q=3,4}$	$[V_1]$
β	$U(n_1+1)$	$[V_1]$	$[V_1]$	$[V_1]$
	$U(n_1)$	$[V_1]$	$[V_1]-[0,1]_{q=3,4}$	$[V_1]$
			$[V_1]-[1,0]_{q=1,2}$	
γ	$U(n_1+1)$	$[V_1]+[0,1]$	$[V_1]+[0,1]$	$[V_1]+[0,1]$
	$U(n_1)$	$[V_1]+[-1,1]$	$[V_1]-[1,0]_{q=1,2}$	$[V_1]$
y: α'	$U(n_2+1)$	$[V_2]+[1,0]$	$[V_2]+[1,0]$	$[V_2]+[1,0]$
	$U(n_2)$	$[V_2]+[1,-1]$	$[V_2]+[1,0]_{q=1,3}$	$[V_2]$
β'	$U(n_2+1)$	$[V_2]+[1,1]$	$[V_2]+[1,1]$	$[V_2]+[1,1]$
	$U(n_2)$	$[V_2]$	$[V_2]+[1,0]_{q=1,3}$	$[V_2]$
			$[V_2]+[0,1]_{q=2,4}$	
γ'	$U(n_2+1)$	$[V_2]+[0,1]$	$[V_2]+[0,1]$	$[V_2]+[0,1]$
	$U(n_2)$	$[V_2]+[-1,1]$	$[V_2]+[0,1]_{q=2,4}$	$[V_2]$

Table 6. Reasonable choices of IRs for type III ($E_L E_R$)

		Level	Final	Intermediate	Initial
x:	α	$U(n_1+1)$	$[V_1]+[1,0]$	$[V_1]+[1,0]$	$[V_1]+[1,0]$
		$U(n_1)$	$[V_1]+[1,-1]$	$[V_1]+[1,0]_{q=1,2}$	$[V_1]$
	β	$U(n_1+1)$	$[V_1]+[1,1]$	$[V_1]+[1,1]$	$[V_1]+[1,1]$
		$U(n_1)$	$[V_1]$	$[V_1]+[0,1]_{q=3,4}$	$[V_1]$
				$[V_1]+[1,0]_{q=1,2}$	
	γ	$U(n_1+1)$	$[V_1]+[0,1]$	$[V_1]+[0,1]$	$[V_1]+[0,1]$
		$U(n_1)$	$[V_1]+[-1,1]$	$[V_1]+[0,1]_{q=3,4}$	$[V_1]$
y:	α'	$U(n_2+1)$	$[V_2]+[1,0]$	$[V_2]+[1,0]$	$[V_2]+[1,0]$
		$U(n_2)$	$[V_2]+[1,-1]$	$[V_2]-[0,1]_{q=2,4}$	$[V_2]$
	β'	$U(n_2+1)$	$[V_2]$	$[V_2]$	$[V_2]$
		$U(n_2)$	$[V_2]$	$[V_2]-[1,0]_{q=1,3}$	$[V_2]$
				$[V_2]-[0,1]_{q=2,4}$	
	γ'	$U(n_2+1)$	$[V_2]+[0,1]$	$[V_2]+[0,1]$	$[V_2]+[0,1]$
		$U(n_2)$	$[V_2]+[-1,1]$	$[V_2]-[1,0]_{q=1,3}$	$[V_2]$

5. Example

In this section, we give an example of type I:

$$U(8) \supset U(4) \times U(4),$$

namely:

$$\left\langle [2^4;1^3];\ \begin{matrix} 1 & 4 \\ 2 & \\ 4 & \\ & \end{matrix} \otimes \begin{matrix} 5 & 6 \\ 6 & 7 \\ 7 & 8 \\ 8 & \end{matrix} \,\middle|\, E_{73}E_{62} \,\middle|\, [2^4,1^3];\ \begin{matrix} 1 & 2 \\ 2 & 4 \\ 3 & \\ 4 & \end{matrix} \otimes \begin{matrix} 5 & 8 \\ 6 & \\ 7 & \\ 8 & \end{matrix} \right\rangle_{\beta\alpha'},$$

in order to illustrate how to use the tables. For this example:

$$d = 4$$

$$d_1' = 3 \qquad d_2' = 2 \qquad \Delta' s = 0 \qquad (\lambda_2^{(2)})' = 3$$

$$d_1 = 3 \qquad d_2 = 4 \qquad \Delta s = 1 \qquad \lambda_2^{(2)} = 1$$

$$x = \beta \qquad y = \alpha'.$$

From Table 3, we see that the two possibilities of pq are $pq = 24$ and $pq = 42$ for $xy = \beta\alpha'$. From Table 4, the IRs corresponding to $y = \alpha'$ for both $pq = 24$ and 42 are:

	final	intermediate	initial
$U(n_2+2)$	[4, 3]	[4, 3]	[4, 3]
$U(n_2+1)$	[4, 3]	[4, 2]	[4, 2]
$U(n_2)$	[4, 3]	[4, 2]	[4, 1]

Table 7. Formulae for F_p, B_q

Type I ($E_L E_L$)

p, q	F_p	B_q
1	$(-1)^{(\lambda_1^{(2)})' + \Delta' s - 1} \sqrt{\frac{(d_1' - \Delta' s)(d_2' - 1 - \Delta' s)}{d_1'(d_2' - 1)}}$	$(-1)^{\lambda_1^{(2)} + \Delta s} \sqrt{\frac{(d_1 - 1 - \Delta s)(d_2 - \Delta s)}{(d_1 - 1)d_2}}$
2	$(-1)^{(\lambda_2^{(2)})' - 1} \sqrt{\frac{(d_1' + d_2' - \Delta' s)(\Delta' s + 1)}{d_1'(d_2' + 1)}}$	$(-1)^{\lambda_2^{(2)}} \sqrt{\frac{(d_1 + d_2 - 1 - \Delta s)\Delta s}{(d_1 - 1)d_2}}$
3	$(-1)^{(\lambda_1^{(2)})' - 1} \sqrt{\frac{(d_1' + d_2' - 1 - \Delta' s)\Delta' s}{d_1'(d_2' - 1)}}$	$(-1)^{\lambda_1^{(2)}} \sqrt{\frac{(d_1 + d_2 - \Delta s)(1 + \Delta s)}{(d_1 + 1)d_2}}$
4	$(-1)^{(\lambda_2^{(2)})' + \Delta' s - 1} \sqrt{\frac{(d_1' - 1 - \Delta' s)(d_2' - \Delta' s)}{d_1'(d_2' + 1)}}$	$(-1)^{\lambda_2^{(2)} + \Delta s} \sqrt{\frac{(d_1 - \Delta s)(d_2 - 1 - \Delta s)}{(d_1 + 1)d_2}}$

Type II ($E_R E_L$). The B_q are the same as the F_p except without prime

p	$F_p(B_q)$
1	$(-1)^{(\lambda_1^{(2)})' + \Delta' s} \sqrt{\frac{(d_1' - 1 - \Delta' s)(d_2' - \Delta' s)}{(d_1' - 1)(d_2' + 1)}}$
2	$(-1)^{(\lambda_2^{(2)})'} \sqrt{\frac{(d_1' + d_2' - 1 - \Delta' s)\Delta' s}{(d_1' - 1)(d_2' - 1)}}$
3	$(-1)^{(\lambda_1^{(2)})'} \sqrt{\frac{(d_1' + d_2' - \Delta' s)(1 + \Delta' s)}{(d_1' + 1)(d_2' + 1)}}$
4	$(-1)^{(\lambda_2^{(2)})' + \Delta' s} \sqrt{\frac{(d_1' - \Delta' s)(d_2' - 1 - \Delta' s)}{(d_1' + 1)(d_2' - 1)}}$

Type III ($E_L E_R$). The B_q are the same as the F_p except without prime

p	$F_p(B_q)$
1	$(-1)^{(\lambda_1^{(2)})' + \Delta' s - 1} \sqrt{\frac{(d_1' - \Delta' s)(d_2' - 1 - \Delta' s)}{(d_1' + 1)(d_2' - 1)}}$
2	$(-1)^{(\lambda_2^{(2)})' - 1} \sqrt{\frac{(d_1' + d_2' - \Delta' s)(\Delta' s + 1)}{(d_1' + 1)(d_2' + 1)}}$
3	$(-1)^{(\lambda_1^{(2)})' - 1} \sqrt{\frac{(d_1' + d_2' - 1 - \Delta' s)\Delta' s}{(d_1' - 1)(d_2' - 1)}}$
4	$(-1)^{(\lambda_2^{(2)})' + \Delta' s - 1} \sqrt{\frac{(d_1' - 1 - \Delta' s)(d_2' - \Delta' s)}{(d_1' - 1)(d_2' + 1)}}$

Table 8. Formulae for C_x, D_y

Type I ($E_L E_L$)			
	$\alpha(\alpha')$	$\beta(\beta')$	$\gamma(\gamma')$
C_x	1	1	1
D_y	1	1	1
Types II ($E_R E_L$) and III ($E_L E_R$)			
	$\alpha(\alpha')$	$\beta(\beta')$	$\gamma(\gamma')$
C_x	$\sqrt{\frac{(d'_1-1)(d_1+1)}{d'_1 d_1}}$	1	$\sqrt{\frac{(d'_1+1)(d_1-1)}{d'_1 d_1}}$
D_y	$\sqrt{\frac{(d'_2-1)(d_2+1)}{d'_2 d_2}}$	1	$\sqrt{\frac{(d'_2+1)(d_2-1)}{d'_2 d_2}}$

The IRs corresponding to $x = \beta$ are:

	final	intermediate	initial
$U(n_1+2)$	$[4, 2]$	$[4, 2]$	$[4, 2]$
$U(n_1+1)$	$[4, 1]_{pq=24}$	$[4, 1]_{pq=24}$	$[4, 2]$
	$[3, 2]_{pq=42}$	$[3, 2]_{pq=42}$	
$U(n_1)$	$[3, 1]$	$[4, 1]_{pq=24}$	$[4, 2]$
		$[3, 2]_{pq=42}$	

From Tables 7 and 8 we obtain:

$$C_\beta = D_{\alpha'} = 1.$$

For $pq = 24$, we get:

$$F_2 = (-1)^{(\lambda_2^{(2)})' - 1} \sqrt{\frac{(d'_1 + d'_2 - \Delta' s)(\Delta' s + 1)}{d'_1(d'_2+1)}}$$

$$B_4 = (-1)^{\lambda_2^{(2)} + \Delta s} \sqrt{\frac{(d_1 - \Delta s)(d_2 - 1 - \Delta s)}{(d_1+1)d_2}}.$$

For $pq = 42$, we get:

$$F_4 = (-1)^{(\lambda_2^{(2)})' + \Delta' s - 1} \sqrt{\frac{(d'_1 - \Delta' s - 1)(d'_2 - \Delta' s)}{d'_1(d'_2+1)}}$$

$$B_2 = (-1)^{\lambda_2^{(2)}} \sqrt{\frac{(d_1 + d_2 - 1 - \Delta s)(\Delta s)}{(d_1-1)d_2}}.$$

So we have:

$$\left\langle [2^4,1^3];\ \begin{array}{ll} 1 & 4 \\ 2 & \\ 4 & \end{array} \otimes \begin{array}{ll} 5 & 6 \\ 6 & 7 \\ 7 & 8 \\ 8 & \end{array} \right| E_{73}E_{62} \left| [2^4,1^3];\ \begin{array}{ll} 1 & 2 \\ 2 & 4 \\ 3 & \\ 4 & \end{array} \otimes \begin{array}{ll} 5 & 8 \\ 6 & \\ 7 & \\ 8 & \end{array} \right\rangle_{\beta\alpha'}$$

$$= A'_{24}\left\langle \begin{array}{ll} 1 & 4 \\ 2 & 6 \\ 4 & \\ 5 & \end{array} \right| E_{53}E_{62} \left| \begin{array}{ll} 1 & 2 \\ 2 & 4 \\ 3 & \\ 4 & \end{array} \right\rangle_{pq=24} \left\langle \begin{array}{ll} 5 & 6 \\ 6 & 7 \\ 7 & 8 \\ 8 & \end{array} \right| E_{7,10}E_{69} \left| \begin{array}{ll} 5 & 8 \\ 6 & 9 \\ 7 & 10 \\ 8 & \end{array} \right\rangle_{pq=24}$$

$$+ A'_{42}\left\langle \begin{array}{ll} 1 & 4 \\ 2 & 5 \\ 4 & \\ 6 & \end{array} \right| E_{53}E_{62} \left| \begin{array}{ll} 1 & 2 \\ 2 & 4 \\ 3 & \\ 4 & \end{array} \right\rangle_{pq=42} \left\langle \begin{array}{ll} 5 & 6 \\ 6 & 7 \\ 7 & 8 \\ 8 & \end{array} \right| E_{7,10}E_{69} \left| \begin{array}{ll} 5 & 8 \\ 6 & 9 \\ 7 & 10 \\ 8 & \end{array} \right\rangle_{pq=42}$$

$$= A'_{24}M^1_{24}M^2_{24} + A'_{42}M^1_{42}M^2_{42}.$$

Substituting all the parameters, combining the basic calculation of two-body operator matrix elements in [6], we obtain:

$$A'_{24} = \frac{\sqrt{5}}{6}$$

$$A'_{42} = -\left(\frac{\sqrt{5}}{3\sqrt{2}}\right)$$

$$M^1_{24} = \frac{2}{\sqrt{3}}$$

$$M^2_{24} = \sqrt{2}$$

$$M^1_{42} = -\frac{1}{\sqrt{6}}$$

$$M^2_{42} = \sqrt{2}$$

Finally:

$$M = \left(\frac{\sqrt{5}}{6}\right)\left(\frac{2}{\sqrt{3}}\right)(\sqrt{2}) + \left(\frac{-\sqrt{5}}{3\sqrt{2}}\right)\left(-\frac{1}{\sqrt{6}}\right)(\sqrt{2}) = \sqrt{\frac{5}{6}}.$$

This result may be checked by the following steps: (1) transform the non-canonical bases into the canonical ones by the subduction coefficients of [15]; (2) calculate the linear combination of two-body matrix elements between the canonical bases in $U(n)$.

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